In the above examples, each segment (i.e., basis function) for operator elements $l_{m,n}$ in (8) and (9) has been subdivided into three subsegments, with weighting approximating a triangle function as suggested by Harrington [6]. In doing so, the $l_{m,n}$ are more accurate and the number of segments can be reduced by one half without an apparent increase in error. In fact, in all the examples above, no more than 30 segments are used on both the discontinuity and the waveguide walls.

For simplicity and for comparison with the available exact solutions, only the fundamental TE_{01} mode has been assumed propagating. It is easy to see, however, that higher TE_{0N} modes can also be assumed to propagate without unduly increasing the computation time. For each additional mode, two field elements f_m, corresponding to the reflection and transmission coefficients of the additional mode, are needed. The extra computation is therefore not much more than required by having two extra segments on the waveguide walls.

Finally, it is to be pointed out that the present method, as well as most other numerical methods, is suitable for treating waveguide with electrically small dimensions. As the guide gets larger, the ray optics method [1]–[4] becomes superior.

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Diffraction of a Wave Beam by an Aperture

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***Abstract*—The diffraction field of a wave beam from a circular and a rectangular aperture is obtained in the Fresnel region by using the Huygens–Kirchhoff approximation. The diffraction field in the Fraunhofer region can be obtained simply by replacing a parameter. The diffraction field is then expanded into a series of beam mode functions.**

From the field distributions and the expansion coefficients, which represent the coupling of the incident beam to the various modes in the diffraction field, the effects of an aperture on the incident beam can be known. With this mode expansion method, the conditions for optimum coupling between fundamental modes are obtained and solved numerically.

I. Introduction

THE output wave beam from optical structures, like Fabry–Perot resonators or optical transmission lines, can be described by Hermite–Gaussian [1] or Laguerre–Gaussian [2] functions.

Apertures, such as irises, are often used as elements of these structures, but there have been few papers that discussed the effects of an aperture on the wave beam. Only the diffraction losses due to the finite sizes of the lens, or mirror apertures that are used as elements of transmission lines [2] or of resonators [3], have been discussed.

The diffraction from an aperture is one of the fundamental problems in electromagnetic field theory and many detailed theories have been compiled for plane wave or spherical wave incidence. The main reason why the diffraction problem for a wave beam has not yet been discussed may be explained by the complexity of the beam wave functions. Up to the present knowledge of the diffraction field of plane waves has been applied to this case.

But, as is well known, if a wave beam of an optical structure is incident on another system, a set of modes of the system is excited or the parameters of the incident wave beam are transformed into different beam parameters [4]. For example, a thin lens transforms these parameters from one set to another. These effects cannot be explained by the analogy of plane wave diffraction.

For this reason, the diffraction problems of a wave beam from a circular and a rectangular aperture are dis-

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cussed here. The diffraction field is obtained by using the Huygens–Kirchhoff formulation.

From the diffraction field, the field variations in the Fresnel region and the half-power angle, which is one of the important parameters, are obtained.

This diffraction field is then expanded into a series of orthogonal beam mode functions. The expansion coefficients represent the beam mode transmission and conversion coefficients or the mode coupling coefficients of the aperture.

It becomes a problem of importance to find out the conditions to make some specified coefficient maximum. In this paper the power coupling coefficient between fundamental modes is considered. This paper treats mainly the diffraction problems for circular geometries, while those for rectangular ones are discussed briefly.

II. Diffraction Field

A. Circular Geometries

When an electromagnetic field $F(r, \theta, z)$ is incident on a plane aperture, which is located at $z=z_0$ as shown in Fig. 1(a), the diffraction field $U(r, \theta, z)$ in the Fresnel region is, by using the scalar Huygens–Kirchhoff approximation, given by [5]

$$U(r,\theta,z) = \frac{jk}{2\pi(z-z_0)} \exp\left[-jk(z-z_0)\right] \cdot \int_{S_0} F(r_0,\theta_0,z_0) \exp\left[-\frac{jk}{2(z-z_0)} \cdot \left\{r^2 + r_0^2 - 2rr_0 \cos(\theta-\theta_0)\right\}\right] r_0 dr_0 d\theta_0 \quad (1)$$

where $k = 2\pi/\lambda$ is the wavenumber of the field and S_0 represents the aperture.

In the near zone, the diffraction field must be calculated rigorously, but for optical waves, which are our main concern, this region is of little importance. Schell and Tyras [6] have used a slightly different diffraction formula. But if the wavelength of the field is small compared with the dimension of a diffracting object, the Huygens–Kirchhoff formula is, as is well known, a good approximation to the rigorous diffraction problem [7].

Let the incident field F be a wave beam ψ_{mn} that is given by [2]

$$\psi_{mn}(r,\theta,z) = \sqrt{\frac{2n!}{\pi\epsilon_m(n+m)!}} \exp\left[-jk(z+z_s)\right] \eta(\eta r)^m L_n^m(\eta^2 r^2) \cdot \exp\left[-\frac{1}{2}\eta^2 r^2 + j(2n+m+1)\tan^{-1}\xi - j\frac{1}{2}\eta^2 \xi r^2\right] \cos(m\theta) \quad (2)$$

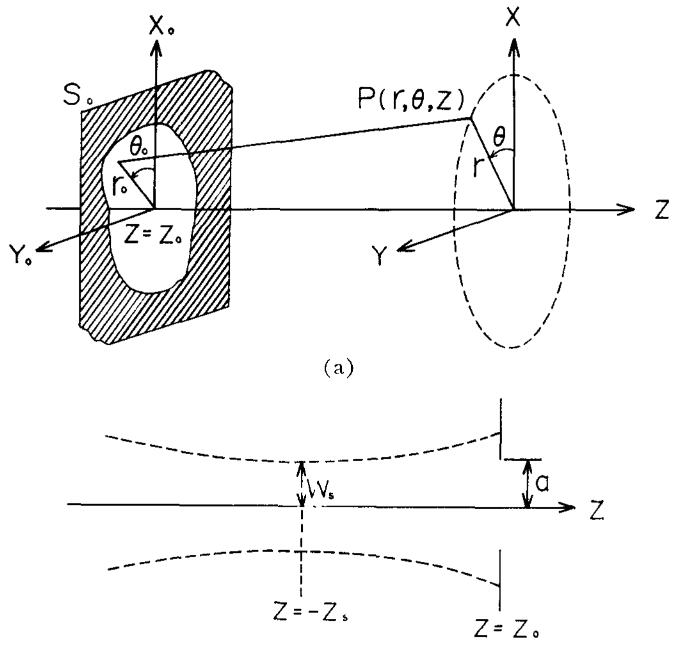

Fig. 1. Geometry of the problem. (a) Coordinate system. (b) Incident wave beam.

where $\epsilon_m = 2$ for $m=0$, $\epsilon_m = 1$ for $m \neq 0$

$$\xi = \frac{2(z+z_s)}{kw_s^2} \quad (3)$$

$$\eta = \frac{\sqrt{2}}{w_s\sqrt{1+\xi^2}} \quad (4)$$

and $L_n^m(x)$ are the generalized Laguerre polynomials defined by

$$L_n^m(x) = \sum_{i=0}^{n} \binom{n+m}{n-i} \frac{(-x)^i}{i!} \quad (5)$$

$\binom{n+m}{n-i} = {}_{n+m}C_{n-i}$ is the binomial coefficient.

This wave beam $\psi_{mn}(r, \theta, z)$ has the smallest spot size w_s at $z = -z_s$ as shown in Fig. 1(b).

The following case is considered here. The beam is incident normally on a circular aperture whose radius is a, and the propagation axis of the incident beam is coincident with the axis that passes through the center of the aperture.

Substituting ψ_{mn} for F in (1), we obtain the corresponding diffraction field U_{mn}, which is, after integrating with respect to θ_0, given by

$$U_{mn}(r,\theta,z) = \sqrt{\frac{2n!}{\pi\epsilon_m(n+m)!}} \exp\left[-jk(z+z_s)\right] \frac{kj^{m+1}}{z-z_0} \cos(m\theta) \cdot \exp\left[j(2n+m+1)\tan^{-1}\xi_0 - \frac{jkr^2}{2(z-z_0)}\right] \cdot \int_0^a (\eta_0 r_0)^{m+1} L_n^m(\eta_0^2 r_0^2) \cdot \exp\left(-\frac{1}{2}\eta_0^2 \tau^2 r_0^2\right) J_m\left(\frac{krr_0}{z-z_0}\right) dr_0 \quad (6)$$

where ξ_0 and η_0 represent the values of ξ and η at the position of the aperture, $J_m(x)$ is the mth-order Bessel function, τ^2 is defined by

$$\tau^2 = \sigma_0^2 + \frac{jk}{\eta_0^2(z - z_0)}, \qquad \sigma_0^2 = 1 + j\xi_0. \tag{7}$$

The integration in (6) can be done most directly by expanding $L_n^m(\eta_0^2 r_0^2)$ and $J_m(krr_0/z-z_0)$ into power series [8]. But here another method is used that is much more convenient for further discussions.

The Bessel function is related to the generalized Laguerre polynomials by [9, p. 189]

$$J_m(2\sqrt{tx}) = \exp(-t) \sum_{p=0}^{\infty} \frac{L_p^m(x)t^p}{(p+m)!} (\sqrt{tx})^m. \tag{8}$$

In this formula, let

$$x = A^2 r^2 \qquad t = \frac{k^2 r_0^2}{4A^2(z - z_0)^2} \tag{9}$$

where A^2 is, in this case, arbitrary and independent of the variable r_0. Then

$$J_m\left(\frac{krr_0}{z - z_0}\right) = \exp\left[-\frac{k^2 r_0^2}{4A^2(z - z_0)^2}\right] \left\{\frac{krr_0}{2(z - z_0)}\right\}^m \cdot \sum_{p=0}^{\infty} \frac{L_p^m(A^2 r^2)}{(p+m)!} \left\{\frac{k^2 r_0^2}{4A^2(z - z_0)^2}\right\}^p. \tag{10}$$

Substituting this into (6), we obtain the diffraction field U_{mn} given by

$$\begin{aligned} U_{mn}(r, \theta, z) &= \sqrt{\frac{2n!}{\pi\epsilon_m(n+m)!}} \exp[-jk(z + z_s)] \frac{kj^{m+1}}{2\eta_0(z - z_0)} \\ &\cdot \cos(m\theta) \exp\left[j(2n + m + 1)\tan^{-1}\xi_0 - \frac{jkr^2}{2(z - z_0)}\right] \\ &\cdot \left\{\frac{kr}{2\eta_0(z - z_0)}\right\}^m \sum_{p=0}^{\infty}\sum_{q=0}^{n} \frac{L_p^m(A^2 r^2)}{(p+m)!} \left\{\frac{k^2}{4A^2\eta_0^2(z - z_0)^2}\right\}^p \\ &\cdot \binom{n+m}{n-q} \frac{(-1)^q(p+q+m)!}{q!} \left(\frac{2}{B}\right)^{p+q+m+1} \\ &\cdot \left[1 - \exp\left(-\frac{1}{2}\eta_0^2 a^2 B\right) \sum_{s=0}^{p+q+m} \frac{1}{s!}\left(\frac{\eta_0^2 a^2 B}{2}\right)^s\right] \end{aligned} \tag{11}$$

where

$$B = \tau^2 + \frac{k^2}{2\eta_0^2 A^2(z - z_0)^2}. \tag{12}$$

This gives us the diffraction field of a wave beam in the Fresnel region from a circular aperture whose radius is a. In this expression the parameter A^2 is included. The diffraction field U_{mn} must be independent of A^2. This can be seen from (10).

To calculate the field numerically this parameter can be chosen arbitrarily. For example, if we let A^2 be large enough, (11) becomes

$$\begin{aligned} U_{mn}(r, \theta, z) &= \sqrt{\frac{2n!}{\pi\epsilon_m(n+m)!}} \exp[-jk(z + z_s)] \frac{kj^{m+1}}{2\eta_0(z - z_0)} \\ &\cdot \cos(m\theta) \exp\left[j(2n + m + 1)\tan^{-1}\xi_0 - \frac{jkr^2}{2(z - z_0)}\right] \\ &\cdot \left\{\frac{kr}{2\eta_0(z - z_0)}\right\}^m \sum_{p=0}^{\infty}\sum_{q=0}^{n} \frac{(-1)^{p+q}(p+q+m)!}{p!\,q!(p+m)!} \\ &\cdot \left\{\frac{kr}{2\eta_0(z - z_0)}\right\}^{2p} \binom{n+m}{n-q} \left(\frac{2}{\tau^2}\right)^{p+q+m+1} \\ &\cdot \left[1 - \exp\left(-\frac{1}{2}\eta_0^2\tau^2 a^2\right) \sum_{s=0}^{p+q+m} \frac{1}{s!}\left(\frac{\eta_0^2\tau^2 a^2}{2}\right)^s\right]. \end{aligned} \tag{13}$$

This coincides with the result obtained by direct expansion of $L_n^m(x)$ and $J_m(x)$ into power series [8].

The result, which includes an infinite series, seems rather troublesome for numerical computations. However, in paraxial cases, which are our main concern, this series converges fast enough.

Fig. 2 shows the distributions of the diffraction field on the propagation axis for fundamental mode incidence ($m = n = 0$). β is the ratio of the aperture radius a to the spot size w_0 of the incident beam at the position of the aperture, that is, $\beta = a/w_0$.

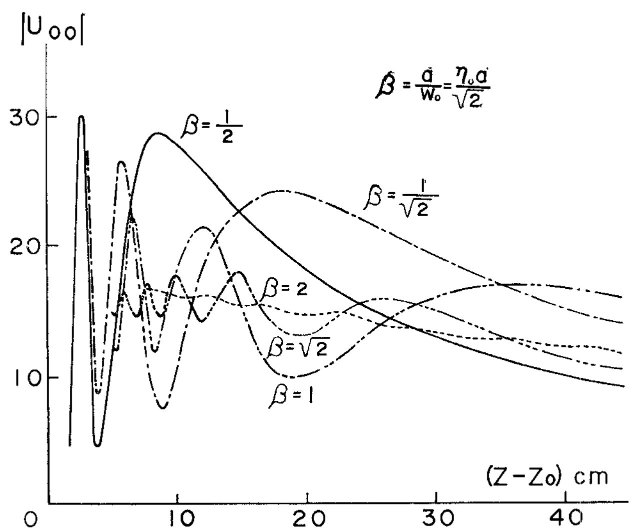

Fig. 2. Field distributions on the propagation axis. The incident beam is the fundamental mode whose parameters are as follows. $\lambda = 6328$ Å, $w_s = 0.71$ mm. The aperture is put at the position of the beam waist. β is the ratio of the aperture radius to the spot size.

When β is larger than 2, the effects of the aperture can hardly be noticed. In other words, if the radius of the aperture is larger than twice the spot size of the incident beam, the effects of the aperture are negligible, as far as the field distributions on the axis are concerned.

When $(z - z_0)$ is large enough the fluctuations on the propagation axis vanish and the field varies in inverse proportion to $(z - z_0)$. This region is regarded as the Fraunhofer region.

The far field is obtained if we use the Fraunhofer ap-

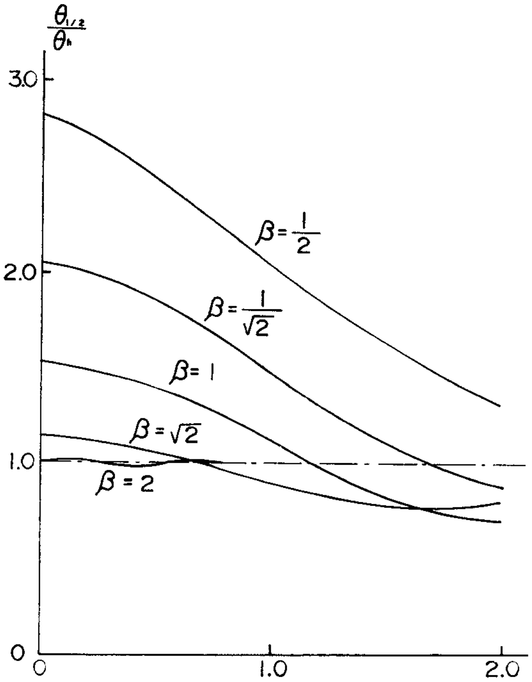

Fig. 3. Half-power angle $\theta_{\frac{1}{2}}$ of the diffraction field for fundamental mode incidence. θ_h is the half-power angle of the incident beam. When k and w_s are fixed, $|\xi_0|$ is proportional to the distance between the positions of the smallest spot size and an aperture. $\beta = a/w_0 =$ const. means the variation of a with ξ_0 because $w_0 = w_s\sqrt{1+\xi_0^2}$.

proximation of the diffraction formula (1). If τ^2 is replaced with σ_0^2, (11) or (13) represents the diffract[illegible] field in the Fraunhofer region.

From the far field, the important half-power angle $\theta_{\frac{1}{2}}$ is obtained numerically. The half-power angle is the full angle in a meridian plane between the two directions in which the power radiated is one-half the maximum value [1].

Fig. 3 shows $\theta_{\frac{1}{2}}$ for fundamental mode incidence normalized with respect to the half-power angle θ_h of the incident beam. In some cases $\theta_{\frac{1}{2}}$ can be smaller than θ_h. This may apparently seem to contradict the diffraction phenomena. But θ_h is defined for the fundamental mode and, as will be shown in Section III, the diffraction field can be considered as the superposition of the fundamental mode and its higher modes. At any specified point, each higher mode may contribute either constructively or destructively to the fundamental mode depending on the incidence configurations.

Therefore, under certain incidence configurations, $\theta_{\frac{1}{2}}$ can be smaller than θ_h. In this case, the far-field spreading angle of the wave field is reduced by an iris, because the diffraction field behind the iris can be regarded as being radiated from a larger aperture than the original field.

This parameter $\theta_{\frac{1}{2}}$ also shows the effects of the aperture on the incident beam. If $\theta_{\frac{1}{2}}$ is approximately equal to θ_h independently of ξ_0, the effect may be almost negligible. This occurs when β is larger than 2, which is in agreement with our earlier comment regarding the field distribution.

B. *Rectangular Geometries*

The method adopted to obtain the diffraction field for circular geometries can be also used in this case. Here only the results of the diffraction field from a square aperture for fundamental mode incidence will be shown.

The incident beam is, in this case, given by [1]

$$\phi_{00}(x, y, z) = \frac{\eta}{\sqrt{\pi}} \exp\left[-jk(z+z_s)\right] \cdot \exp\left[-\tfrac{1}{2}\eta^2(1+j\xi)(x^2+y^2) + j\tan^{-1}\xi\right] \quad (14)$$

where ξ and η are defined by (3) and (4), respectively.

The diffraction field is obtained similarly as in the case of circular ones, if the formula for rectangular geometries is used. The result is given by

$$V_{00}(x, y, z) = \frac{j}{\lambda\sqrt{\pi}} \frac{\eta_0}{z-z_0} \exp\left[-jk(z+z_s)\right] \cdot \exp\left[j\tan^{-1}\xi_0 - \frac{jk}{2(z-z_0)}(x^2+y^2)\right] g(x,z)g(y,z) \quad (15)$$

where

$$g(u, z) = \frac{2}{\eta_0\tau} \sum_{t=0}^{\infty} H_{2t}\left[\frac{jku}{\eta_0\tau(z-z_0)}\right] \frac{(\eta_0\tau a)^{2t+1}}{(2t+1)!} \quad (16)$$

and $2a$ is the length of the side of the aperture. $H_m(x)$ are the Hermite polynomials defined by

$$H_m(x) = \sum_{t=0}^{[m/2]} \frac{(-1)^t m!}{2^t t!(m-2t)!} x^{m-2t}. \quad (17)$$

The behaviors of this diffraction field are quite similar to those of circular geometries [10].

III. Mode Coupling Coefficients

When a wave beam of a certain optical structure is incident on another system, a set of modes of this latter system is excited. Let the diffraction field for the incident beam ψ_{mn} of unit power be represented as a superposition of $C_{mn}{}^{mn}\mathfrak{\psi}_{mn}$. The set $\{\mathfrak{\psi}_{mn}\}$ may not, in general, be coincident with $\{\psi_{mn}\}$. The beam parameters of $\{\mathfrak{\psi}_{mn}\}$ are denoted as $\boldsymbol{w}_s$, $\boldsymbol{z}_s$, etc. The complex amplitudes $\{C_{mn}{}^{mn}\}$ are defined as the coupling coefficients [4].

To evaluate these coefficients, the field distributions in a certain plane, where z is constant, are equated. In our case, this can be written as follows,

$$U_{mn}(r, \theta, z) = \sum_{m,n} C_{mn}{}^{mn}\mathfrak{\psi}_{mn}(r, \theta, z). \quad (18)$$

Using the orthonormality of $\{\mathfrak{\psi}_{mn}\}$, we obtain the coefficients given by

$$C_{mn}^{\mathbf{m}\mathbf{n}} = \int_0^\infty \int_0^{2\pi} U_{mn}(r, \theta, z)\psi_{\mathbf{m}\mathbf{n}}^*(r, \theta, z)\, r dr d\theta \quad (19)$$

where the asterisk denotes the complex conjugate.

Andrade and Thomas [11] calculated this coefficient without taking diffraction effects into considerations, while Nemoto and Makimoto [12] have proposed the method of calculating the coefficient without obtaining the analytical form of the diffraction field.

Here, for the consistency of the theory, (11) is used for the calculation of (19). If we substitute (11) into (19), the coefficients are given by

$$C_{mn}^{\mathbf{m}\mathbf{n}} = \sqrt{\frac{2n!}{(n+m)!}}\sqrt{\frac{2\mathbf{n}!}{(\mathbf{n}+m)!}}\frac{kj^{m+1}}{2\eta_0(z-z_0)}$$
$$\cdot \exp\left[-jk(z_s - \mathbf{z}_s) + j(2n+m+1)\tan^{-1}\xi_0 - j(2\mathbf{n}+m+1)\tan^{-1}\boldsymbol{\xi}\right]$$
$$\cdot \sum_{p=0}^{\infty}\sum_{q=0}^{n}\frac{(-1)^q(p+q+m)!}{(p+m)!q!}\binom{n+m}{n-q}$$
$$\cdot\left(\frac{2}{B}\right)^{p+q+m+1}\left\{\frac{k^2}{4A^2\eta_0^2(z-z_0)^2}\right\}^p$$
$$\cdot\left[1-\exp\left(-\frac{1}{2}\eta_0^2a^2B\right)\sum_{s=0}^{p+q+m}\frac{1}{s!}\left(\frac{\eta_0^2a^2B}{2}\right)^s\right]\int_0^\infty\left\{\frac{kr}{2\eta_0(z-z_0)}\right\}^m(\mathbf{n}r)^{m+1}$$
$$\cdot\exp\left(-\frac{1}{2}\mathbf{n}^2Cr^2\right)L_p^m(A^2r^2)L_{\mathbf{n}}^m(\mathbf{n}^2r^2)\, dr \quad (20)$$

for $m = \mathbf{m}$, and

$$C_{mn}^{\mathbf{m}\mathbf{n}} = 0 \quad (21)$$

for $m \neq \mathbf{m}$. Here C is defined by

$$C = 1 - j\xi + \frac{jk}{\mathbf{n}^2(z-z_0)}\,. \quad (22)$$

The integral in (20) can be rewritten as follows,

$$\int_0^\infty\left\{\frac{kr}{2\eta_0(z-z_0)}\right\}^m(\mathbf{n}r)^{m+1}$$
$$\cdot\exp\left(-\frac{1}{2}\mathbf{n}^2Cr^2\right)L_p^m(A^2r^2)L_{\mathbf{n}}^m(\mathbf{n}^2r^2)\, dr$$
$$= \frac{1}{2\mathbf{n}}\left\{\frac{k}{2\eta_0\mathbf{n}(z-z_0)}\right\}^m\left(\frac{2}{C}\right)^{m+1}$$
$$\cdot\int_0^\infty \exp(-x)x^mL_p^m\left(\frac{2A^2x}{\mathbf{n}^2C}\right)L_{\mathbf{n}}^m\left(\frac{2x}{C}\right)dx. \quad (23)$$

We choose the parameter A^2 as follows,

$$A^2 = \tfrac{1}{2}C\mathbf{n}^2 \quad (24)$$

and use the formula [13]

$$L_n^m(xy) = \sum_{t=0}^{n}\binom{n+m}{t}(1-x)^tx^{n-t}L_{n-t}^m(y) \quad (25)$$

then the coupling coefficients are given by

$$C_{mn}^{\mathbf{m}\mathbf{n}} = \sqrt{\frac{n!}{(n+m)!}}\sqrt{\frac{\mathbf{n}!}{(\mathbf{n}+m)!}}\exp\left[-jk(z_s-\mathbf{z}_s) + j(2n+m+1)\tan^{-1}\xi_0 + j(m+1)\frac{\pi}{2} - j(2\mathbf{n}+m+1)\tan^{-1}\boldsymbol{\xi}\right]$$
$$\cdot\sum_{p=0}^{\mathbf{n}}\sum_{q=0}^{n}\frac{(-1)^q(p+q+m)!}{p!q!}\binom{n+m}{n-q}$$
$$\cdot\binom{\mathbf{n}+m}{\mathbf{n}-p}\left(\frac{2}{B}\right)^{p+q+m+1}\left(\frac{2}{C}\right)^{p+m+1}$$
$$\cdot\left(\frac{C-2}{C}\right)^{\mathbf{n}-p}\left\{\frac{k^2}{4\eta_0^2A^2(z-z_0)^2}\right\}^p$$
$$\cdot\left\{\frac{k}{2\eta_0\mathbf{n}(z-z_0)}\right\}^{m+1}\left[1-\exp\left(-\frac{1}{2}\eta_0^2a^2B\right)\right.$$
$$\left.\cdot\sum_{s=0}^{p+q+m}\frac{1}{s!}\left(\frac{\eta_0^2a^2B}{2}\right)^s\right]. \quad (26)$$

Using the following relations,

$$C = \frac{k}{\mathbf{n}_0\mathbf{n}(z-z_0)}\exp\left[-j\tan^{-1}\left\{\boldsymbol{\xi} - \frac{k}{\mathbf{n}^2(z-z_0)}\right\}\right] \quad (27)$$

$$\tan^{-1}\boldsymbol{\xi} - \tan^{-1}\left\{\boldsymbol{\xi} - \frac{k}{\mathbf{n}^2(z-z_0)}\right\}$$
$$= \tan^{-1}\frac{k}{\mathbf{n}^2(z-z_0)(1+\boldsymbol{\xi}^2) - k\boldsymbol{\xi}} = -\tan^{-1}\frac{1}{\boldsymbol{\xi}_0} \quad (28)$$

we finally obtain the coupling coefficients given by

$$C_{mn}^{\mathbf{m}\mathbf{n}} = \sqrt{\frac{n!}{(n+m)!}}\sqrt{\frac{\mathbf{n}!}{(\mathbf{n}+m)!}}\exp\left[-jk(z_s-\mathbf{z}_s) + j(2n+m+1)\tan^{-1}\xi_0 - j(2\mathbf{n}+m+1)\tan^{-1}\boldsymbol{\xi}_0\right]$$
$$\cdot\sum_{p=0}^{\mathbf{n}}\sum_{q=0}^{n}\frac{(-1)^{p+q}(p+q+m)!}{p!q!}\binom{n+m}{n-q}$$
$$\cdot\binom{\mathbf{n}+m}{\mathbf{n}-p}\left(\frac{2}{B}\right)^{p+q+m+1}\left(\frac{\mathbf{n}_0}{\eta_0}\right)^{2p+m+1}$$
$$\cdot\left[1-\exp\left(-\frac{1}{2}\eta_0^2a^2B\right)\right.$$
$$\left.\cdot\sum_{s=0}^{p+q+m}\frac{1}{s!}\left(\frac{\eta_0^2a^2B}{2}\right)^s\right]. \quad (29)$$

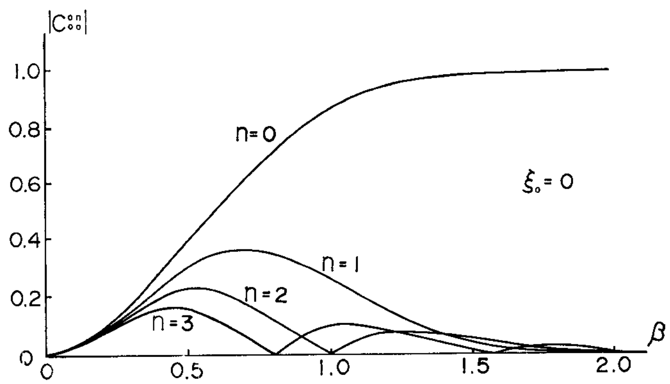

Fig. 4. Transmission and conversion coefficients for fundamental mode incidence. The aperture is put at the position of the beam waist.

These coefficients are independent of z. To see this we use (12) and (24), then the parameter B is given by

$$B = \frac{\eta_0^2 + \mathbf{n}_0^2}{\eta_0^2} + j\frac{\xi_0\eta_0^2 - \boldsymbol{\xi}_0\mathbf{n}_0^2}{\eta_0^2}. \qquad (30)$$

This shows that B is independent of z; therefore, the coupling coefficients given by (29) are also independent of z.

If the sets $\{\psi_{mn}\}$ and $\{\boldsymbol{\psi}_{mn}\}$ are identical, namely, if $\mathbf{z}_s = z_s$ and $\mathbf{w}_s = w_s$, the coupling coefficients represent the transmission and the conversion coefficients of the aperture. In this case, the coefficients are written as $C_{mn}{}^{mn}$.

Fig. 4 shows some of these coefficients for fundamental mode incidence. When β is larger than 2 the mode conversion coefficients $C_{00}{}^{01}$, $C_{00}{}^{02}$, $C_{00}{}^{03}$, etc., are almost equal to zero, while the transmission coefficient $C_{00}{}^{00}$ is approximately equal to unity. This is the most convincing proof of the fact that for β larger than 2 the effects of the aperture are almost negligible.

For rectangular geometries the coupling coefficients are, for fundamental mode incidence, given by

$$\mathbf{C}_{00}{}^{mn} = \mathbf{C}_0{}^m\mathbf{C}_0{}^n \qquad (31)$$

where $\mathbf{C}_0{}^m$ are given as follows,

$$\mathbf{C}_0{}^m = \frac{\sqrt{(2t)!}}{2^t}\sqrt{\frac{2}{B}}\sqrt{\frac{\mathbf{n}_0}{\eta_0}}\exp\left[-jk\frac{1}{2}(z_s - \mathbf{z}_s) - jt\pi + j\frac{1}{2}\tan^{-1}\xi_0 - j\left(2t+\frac{1}{2}\right)\tan^{-1}\boldsymbol{\xi}_0\right] \cdot\left[\sum_{p=0}^{t}\sum_{q=0}^{p}K_{pq} + \sum_{p=t+1}^{\infty}\sum_{q=p-t}^{p}K_{pq}\right] \qquad (32)$$

$$K_{pq} = \sqrt{\frac{2}{\pi}}\frac{(-1)^p(2\mathbf{n}_0a)^{2p-2q}(\sqrt{\eta_0^2a^2B})^{2p+1}}{(2p+1)2^qq!(2p-2q)!(t+q-p)!} \qquad (33)$$

for $\mathbf{m} = 2t$, and

$$\mathbf{C}_0{}^m = 0 \qquad (34)$$

for $\mathbf{m} = 2t+1$. These coefficients do not depend upon z either.

IV. Some Considerations on the Power Coupling Between Fundamental Modes

The coupling coefficients obtained in the previous section are now applied to the following problem. For given position and radius of the aperture, and incident wave beam, how can we determine the parameters $\mathbf{w}_s$ and $\mathbf{z}_s$ of the set $\{\boldsymbol{\psi}_{mn}\}$ to maximize some specified coupling coefficient?

In order to simplify the calculations, we discuss here the power coupling coefficient between the fundamental modes. This coefficient is given by

$$|\mathbf{C}_{00}{}^{00}|^2 = \frac{4\eta_0^2a^2\mathbf{n}_0^2a^2}{(\eta_0^2a^2 + \mathbf{n}_0^2a^2)^2 + (\xi_0\eta_0^2a^2 - \boldsymbol{\xi}_0\mathbf{n}_0^2a^2)^2} \cdot\left[1 + \exp(-\eta_0^2a^2 - \mathbf{n}_0^2a^2) - 2\exp\left\{-\tfrac{1}{2}(\eta_0^2a^2 + \mathbf{n}_0^2a^2)\right\} \cdot\cos\tfrac{1}{2}(\xi_0\eta_0^2a^2 - \boldsymbol{\xi}_0\mathbf{n}_0^2a^2)\right] \qquad (35)$$

for circular geometries, and for rectangular geometries it is

$$|\mathbf{C}_{00}{}^{00}|^2 = \frac{4\eta_0^2a^2\mathbf{n}_0^2a^2}{(\eta_0^2a^2 + \mathbf{n}_0^2a^2)^2 + (\xi_0\eta_0^2a^2 - \boldsymbol{\xi}_0\mathbf{n}_0^2a^2)^2} \cdot[G^2G^{*2}] \qquad (36)$$

where

$$G = 2\Phi\left[\sqrt{(\eta_0^2a^2 + \mathbf{n}_0^2a^2) - j(\xi_0\eta_0^2a^2 - \boldsymbol{\xi}_0\mathbf{n}_0^2a^2)}\right] \qquad (37)$$

$\Phi(x)$ is the error function defined by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \exp(-\tfrac{1}{2}t^2)\,dt. \qquad (38)$$

In (35) and (36) the coefficients of the brackets coincide with the coupling coefficients obtained by Kogelnik [4] for optical modes when the aperture is infinitely large. If we let a be infinite, both brackets become unity. Therefore, they show the effects of the finite aperture.

To maximize these expressions, the following conditions must be satisfied:

$$\xi_0\eta_0^2a^2 = \boldsymbol{\xi}_0\mathbf{n}_0^2a^2 \qquad (39)$$

$$(\eta_0^2a^2 - \mathbf{n}_0^2a^2)\left[1 - \exp\left\{-\tfrac{1}{2}(\eta_0^2a^2 + \mathbf{n}_0^2a^2)\right\}\right] + \mathbf{n}_0^2a^2(\eta_0^2a^2 + \mathbf{n}_0^2a^2)\exp\left[-\tfrac{1}{2}(\eta_0^2a^2 + \mathbf{n}_0^2a^2)\right] = 0 \qquad (40)$$

for circular geometries and

$$\xi_0\eta_0^2a^2 = \boldsymbol{\xi}_0\mathbf{n}_0^2a^2 \qquad (41)$$

$$(\eta_0^2a^2 - \mathbf{n}_0^2a^2)\sqrt{\frac{\pi}{2}}\,\Phi\left(\sqrt{\eta_0^2a^2 + \mathbf{n}_0^2a^2}\right)\sqrt{\eta_0^2a^2 + \mathbf{n}_0^2a^2} + \mathbf{n}_0^2a^2(\eta_0^2a^2 + \mathbf{n}_0^2a^2)\exp\left[-\tfrac{1}{2}(\eta_0^2a^2 + \mathbf{n}_0^2a^2)\right] = 0 \qquad (42)$$

for rectangular geometries. The uniqueness of the solutions of these equations can be proved analytically for (39) and (40), and numerically for (41) and (42). Fig. 5 shows the solutions of both of them.

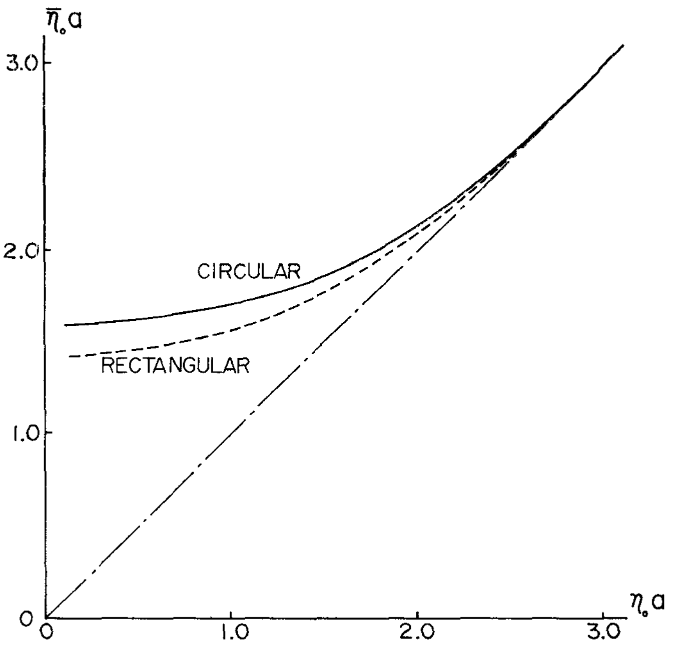

Fig. 5. Solutions for (40) and (42). (Overbar in figure same as bold face in text.)

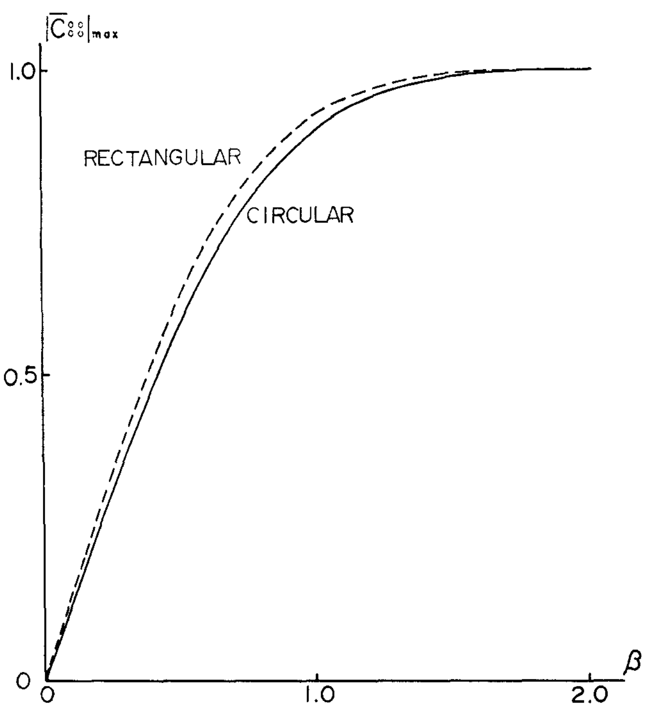

Fig. 6. Maximum power coupling coefficients between fundamental modes. (Overbar in figure same as bold face in text.)

From this result, the optimal values of beam parameters w_s and z_s are obtained from the following expressions:

$$\frac{w_s}{a} = \frac{\sqrt{2}}{\mathbf{n}_0 a\sqrt{1 + \xi_0^2}} \tag{43}$$

$$z_s = \frac{k\xi_0 a^2}{\mathbf{n}_0^2 a^2(1 + \xi_0^2)} - z_0. \tag{44}$$

The maximum coupling coefficients are shown in Fig. 6.

Further discussions of this problem will be applied to the problems of resonators with an internal aperture [13] and of the transmission lines in which lenses and apertures are used.

V. Conclusions

The diffraction field of a wave beam from a plane aperture is obtained. From its intensity distributions on the propagation axis the diffraction effects of an aperture on the wave beam can be almost known.

The mode expansion method, however, provides us with more theoretical and quantitative knowledge of the effects. The results show that the effects are negligible if the aperture radius is larger than twice the spot size of the incident beam, and this result is also valid for rectangular geometries.

In this paper only one of the problems that could be treated with the mode expansion method is considered. The expansion coefficients are also useful for the analysis of optical structures.

Investigations of the case where the propagation axis and the axis that passes through the center of the aperture do not coincide are also of practical importance. This problem can be solved in the same way.

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